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LETTER TO THE EDITOR

Position space renormalisation group study of shortest paths in percolation clusters

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Abstract. The average length of the shortest paths along the backbone of the incipient infinite cluster is expected to diverge near the percolation threshold p_c as $|p-p_c|^{-\nu_s}$, with ν_s different from the correlation length exponent ν. We have applied here position space renormalisation group (PSRG) technique to obtain ν_s both for square and cubic lattices and for triangular lattice in two dimensions. The results obtained ($\nu_s = 1.49$ for $d=2$ and 1.21 for $d=3$) are in very good agreement with the existing results obtained by different methods.

In the percolation problem, one generally talks of the connections or the nature of the connections in a lattice, in which bonds or sites have been removed randomly. For instance, one might want to find the probability that two sites i and j are connected through the bonds. However, apart from that, one might want to find the shortest path between the sites i and j. The mean length of all the shortest paths between all pairs of points over the backbone cluster is expected to diverge at the percolation threshold p_c as $|p-p_c|^{-\nu_s}$, with ν_s different from the correlation length exponent ν_p. The behaviour of this shortest path near p_c has already been studied employing different techniques. The fractal dimension of the shortest connecting paths in the backbone cluster has been studied numerically by Herrmann *et al* (1984), who obtained $D_E = 1.10 \pm 0.05$ and 1.35 ± 0.05 for the fractal dimensions of the shortest connecting paths in two and three dimensions respectively. These results ($D_E = \nu_s/\nu_b$) are in fair agreement with the results obtained employing series expansion ($\nu_s = 1.38 \pm 0.10$ for $d=2$ and 1.18 ± 0.07 for $d=3$, ζ_{min} of Hong and Stanley 1983a) and PSRG ($\nu_s = 1.55$ for $d=2$, ζ_{min} of Hong and Stanley 1983b) techniques. Here, we have applied the same PSRG techniques as was used by Hong and Stanley (1983b) to obtain ν_s for square (both for the scale factors $b=2$ and $b=3$ and extrapolated), two-dimensional triangular and cubic lattices and the results obtained are found to be in very good agreement with the previous results.

To apply the renormalisation group technique on the bond-diluted square lattice, we chose a cell as shown in figure 1 that covers the lattice with bonds and rescales to another bond. The occupation probability p' of a renormalised bond is obtained in terms of the occupation probability p of a bond in the original lattice and we have, for the scaling factor $b=2$,

$$p' = p^5 + 5p^4(1-p) + 8p^3(1-p)^2 + 2p^2(1-p)^3 \tag{1}$$

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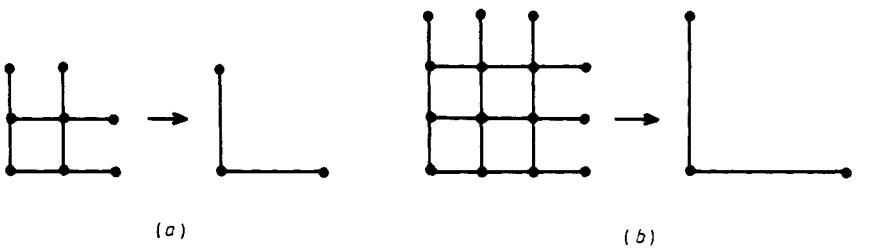

Figure 1. Elementary cell used for bond percolation on a square lattice. (*a*) The cell for the scale factor $b=2$ and (*b*) the cell for the scale factor $b=3$. After renormalisation the cell becomes a vertical and a horizontal bond.

and for $b=3$

$$p'=p^{13}+13p^{12}(1-p)+78p^{11}(1-p)^2+283p^{10}(1-p)^3+677p^9(1-p)^4 +1078p^8(1-p)^5+1089p^7(1-p)^6+627p^6(1-p)^7 +209p^5(1-p)^8+38p^4(1-p)^9+3p^3(1-p)^{10}. \tag{2}$$

To find the critical behaviour of the shortest path, we associate a quantity x with the length of each bond, in terms of which for a certain bond occupation configuration, the shortest path required to traverse the cell can be determined. For $b=2$, when all the bonds are present, the shortest path to traverse the cell is $2x$ and for $b=3$, it is $3x$. The renormalised shortest path length x' for the renormalised bond is thus obtained in terms of x, p, and p' and our result for $b=2$ is

$$p'x'=2xp^5+10xp^4(1-p)+18xp^3(1-p)^2+4xp^2(1-p)^3 \tag{3}$$

and for $b=3$

$$p'x'=3xp^{13}+39xp^{12}(1-p)+234xp^{11}(1-p)^2+873xp^{10}(1-p)^3+2195xp^9(1-p)^4 +3696xp^8(1-p)^5+3865xp^7(1-p)^6+2215xp^6(1-p)^7 +711xp^5(1-p)^8+122xp^4(1-p)^9+9xp^3(1-p)^{10}. \tag{4}$$

The non-trivial percolation fixed points p^* found from equations (1) and (2) are listed in table 1. As expected they are the same for $b=2$ and $b=3$ (Bernasconi 1978). Linearising equations (1) and (2) around the non-trivial fixed point p^*, one obtains the correlation length exponent

$$\nu_b=\ln b/\ln(\partial p'/\partial p)|_{p^*}.$$

From equations (3) and (4), one obtains the exponent for the shortest path as

$$\nu_s=\ln(\partial x'/\partial x|_{p^*})\nu_p/\ln b.$$

The values of p^*, ν_p and ν_s for $b=2$ and $b=3$ as well as the extrapolated result using the relations (Eschbach *et al* 1981)

$$1/\nu_b=1/\nu+a/\ln b$$

$$\nu_s=\nu'+c/\ln b$$

for the bond-diluted square lattice have been listed in table 1.

Table 1. Fixed points and exponents for the bond-diluted square, triangular and cubic lattices.

Lattice type	b	p^*	ν_p	ν_x	ν	ν'
Square	2	0.50[a] 0.50[b]	1.43[a]	1.55[a]	1.28[a]	1.49[a]
	3	0.50[a] 0.50[b]	1.37[a]	1.53[a]	$\frac{4}{3}$[b]	1.47 ± 0.07[c] 1.55[d]
Triangular	$\sqrt{3}$	0.50[a] 0.35[b]	1.13[a]	1.55[a]	1.13[a] $\frac{4}{3}$[b]	
Cubic	2	0.21[a] 0.25[b]	1.02[a]	1.21[a]	1.02[a] 0.85[b]	1.21[a] 1.19 ± 0.05[c] 1.18 ± 0.07[d]

[a] This work.
[b] See, e.g., Reynolds *et al* (1980), den Nijs (1979) and Yeomans and Stinchcombe (1979).
[c] See Herrmann *et al* (1984).
[d] See Hong and Stanley (1983a, b).

For the two-dimensional triangular lattice, the cell chosen (Yeomans and Stinchcombe 1979) has been shown in figure 2. In this case we have obtained for $b=\sqrt{3}$

$$p' = p^5 + 5p^4(1-p) + 8p^3(1-p)^2 + 2p^2(1-p)^3 \tag{5}$$

and

$$p'x' = 2xp^5 + 10xp^4(1-p) + 18xp^3(1-p)^2 + 4xp^2(1-p)^3. \tag{6}$$

The results have been listed in table 1. The value of ν_s is found to be in reasonably good agreement with the previous results (for the next cell, for which $b=2\sqrt{3}$, we have obtained $p^*=0.598$, $\nu_b=1.692$ and $\nu_s=2.135$, which deviates greatly from the expected convergence and has not, therefore, been used for extrapolation).

In the case of the three-dimensional basic cell for which $b=2$, we have

$$\begin{aligned} p' = {} & p^{12} + 12p^{11}(1-p) + 66p^{10}(1-p)^2 + 220p^9(1-p)^3 + 493p^8(1-p)^4 \\ & + 776p^7(1-p)^5 + 856p^6(1-p)^6 + 616p^5(1-p)^7 \\ & + 238p^4(1-p)^8 + 48p^3(1-p)^9 + 4p^2(1-p)^{10} \end{aligned} \tag{7}$$

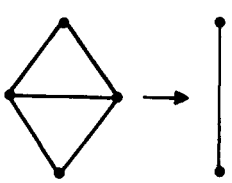

Figure 2. Elementary cell used for bond percolation on a triangular lattice. The scale factor $b=\sqrt{3}$. After renormalisation the cell becomes a vertical bond.

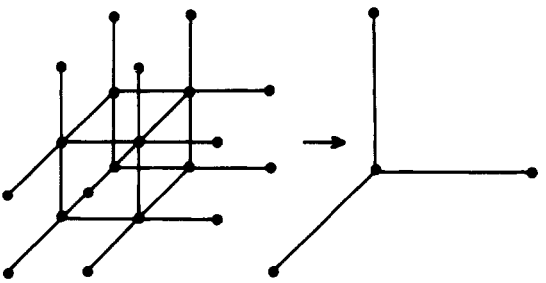

Figure 3. Elementary cell used for bond percolation on a cubic lattice. The scale factor $b=2$. After renormalisation the cell becomes three mutually perpendicular bonds.

and

$$p'x' = 2xp^{12} + 24xp^{11}(1-p) + 132xp^{10}(1-p)^2 + 440xp^9(1-p)^3 + 1000xp^8(1-p)^4 + 1632xp^7(1-p)^5 + 1912xp^6(1-p)^6 + 1464xp^5(1-p)^7 + 548xp^4(1-p)^8 + 104xp^3(1-p)^9 + 8xp^2(1-p)^{10}. \quad (8)$$

The values of p^*, ν_b and ν_s obtained from these two relations have been given in table 1.

In table 1, ν_s has been compared with the ζ_{min} of Hong and Stanley and with $D_E\nu_b$ obtained from Herrmann *et al* (1984), and found to agree very well (both in two and three dimensions) with these results.

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References

Bernasconi J 1978 *Phys. Rev.* B **18** 2185
den Nijs M P M 1979 *J. Phys. A: Math. Gen.* **12** 1857
Eschbach P D, Stauffer D and Herrmann H J 1981 *Phys. Rev.* B **23** 422
Herrmann H J, Hong D C and Stanley H E 1984 *J. Phys. A: Math. Gen.* **17** L261
Hong D C and Stanley H E 1983a *J. Phys. A: Math. Gen.* **16** L475
—— 1983b *J. Phys. A: Math. Gen.* **16** L525
Reynolds P J, Stanley H E and Klein W 1980 *Phys. Rev.* B **21** 1223
Yeomans J M and Stinchcombe R B 1979 *J. Phys. C: Solid State Phys.* **12** 347